AN INTRODUCTION TO DISTRIBUTIONS AND CURRENTS

VI Escuela Doctoral PUCP-UVa

7-17 May 2013

M. G. Soares - UFMG

MANIFOLDS

- A complex manifold (C^k, C^∞, C^ω = real analytic) of dimension n is a topological space M, which is Hausdorff, connected and with a countable basis, endowed with an analytic structure defined as follows: there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ of M and homeomorphisms $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$ where $V_\alpha \subset \mathbb{C}^n$ ($V_\alpha \subset \mathbb{R}^n$) is open, such that the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic (C^k, C^∞, C^ω) where defined. φ_α is called a *chart* and, for $z \in M$, $\varphi_\alpha(z) = (z_1^\alpha, \dots, z_n^\alpha) \in \mathbb{C}^n$ are called the *local coordinates* in U_α. The collection $\{U_\alpha, \varphi_\alpha\}$ is called a holomorphic (C^k, C^∞, C^ω) *atlas* for M.

MANIFOLDS

- If M has dimension n, a connected subset $N \subset M$ is a *submanifold* of dimension $m \leq n$ if, for each $z \in N$ there exists a chart $\{U_\alpha, \varphi_\alpha\}$, with $z \in U_\alpha$, such that φ_α is a homeomorphism between $U_\alpha \cap N$ and an open set of $\mathbb{C}^m \times \{0\} \subset \mathbb{C}^m \times \mathbb{C}^{n-m} \cong \mathbb{C}^n$.

MANIFOLDS

- If M has dimension n, a connected subset $N \subset M$ is a *submanifold* of dimension $m \leq n$ if, for each $z \in N$ there exists a chart $\{U_\alpha, \varphi_\alpha\}$, with $z \in U_\alpha$, such that φ_α is a homeomorphism between $U_\alpha \cap N$ and an open set of $\mathbb{C}^m \times \{0\} \subset \mathbb{C}^m \times \mathbb{C}^{n-m} \cong \mathbb{C}^n$.
- Given manifolds M and N, a map $f : M \longrightarrow N$ is *holomorphic* $(C^k, C^\infty,\ C^\omega)$ provided the compositions $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ are holomorphic $(C^k, C^\infty,\ C^\omega)$ where defined, with ψ_β and φ_α charts in N and M respectively.

MANIFOLDS

- If M has dimension n, a connected subset $N \subset M$ is a *submanifold* of dimension $m \leq n$ if, for each $z \in N$ there exists a chart $\{U_\alpha, \varphi_\alpha\}$, with $z \in U_\alpha$, such that φ_α is a homeomorphism between $U_\alpha \cap N$ and an open set of $\mathbb{C}^m \times \{0\} \subset \mathbb{C}^m \times \mathbb{C}^{n-m} \cong \mathbb{C}^n$.
- Given manifolds M and N, a map $f : M \longrightarrow N$ is *holomorphic* (C^k, C^∞, C^ω) provided the compositions $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ are holomorphic (C^k, C^∞, C^ω) where defined, with ψ_β and φ_α charts in N and M respectively.
- $X \subset M$ is an analytic set if, for each $z \in M$ there is an open neighborhood $U \subset M$ of z and a holomorphic map $f : U \longrightarrow \mathbb{C}^\ell$ such that $X \cap U = f^{-1}(0)$ (ℓ may depend on z).

MANIFOLDS

- If $W \subset M$ is open and $\ell \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ then $C^\ell(W, \mathbb{C})$ ($C^\ell(W, \mathbb{R})$) is the space of functions of class C^ℓ on W. In case W is not open, it is the space of functions which admit a C^ℓ extension to a neighborhood of W.

MANIFOLDS

- If $W \subset M$ is open and $\ell \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ then $C^\ell(W, \mathbb{C})$ ($C^\ell(W, \mathbb{R})$) is the space of functions of class C^ℓ on W. In case W is not open, it is the space of functions which admit a C^ℓ extension to a neighborhood of W.
- Tangent space. Given $z \in M$, write

$$\varphi_\alpha(z) = (z_1^\alpha, \ldots, z_n^\alpha) = \\ = (x_1^\alpha + \mathrm{i}y_1^\alpha, \ldots, x_n^\alpha + \mathrm{i}y_n^\alpha) = \\ (x_1^\alpha, y_1^\alpha, \ldots, x_n^\alpha, y_n^\alpha).$$

- Note that M is naturally a real analytic manifold of dimension $2n$.

MANIFOLDS

- The *real tangent space* of M at z, T_zM is, by definition, the space of differential operators $\nu : C^1(U, \mathbb{R}) \to \mathbb{R}$, where $z \in U \subset M$ is open satisfying (ν is called a tangent vector): (i) ν is $\mathbb{R}$-linear and
  (ii) $\nu(fg) = g(z)\nu(f) + f(z)\nu(g)$.

MANIFOLDS

- The *real tangent space* of M at z, T_zM is, by definition, the space of differential operators $\nu : C^1(U, \mathbb{R}) \to \mathbb{R}$, where $z \in U \subset M$ is open satisfying (ν is called a tangent vector): (i) ν is $\mathbb{R}$-linear and (ii) $\nu(fg) = g(z)\nu(f) + f(z)\nu(g)$.
- By definition, $\dfrac{\partial f}{\partial x_i^\alpha}(z) = \dfrac{\partial(f \circ \varphi_\alpha)}{\partial x_i^\alpha}(\varphi_\alpha(z))$ and similarly for the y_i^αs.

MANIFOLDS

- The *real tangent space* of M at z, T_zM is, by definition, the space of differential operators $\nu : C^1(U, \mathbb{R}) \to \mathbb{R}$, where $z \in U \subset M$ is open satisfying (ν is called a tangent vector): (i) ν is $\mathbb{R}$-linear and (ii) $\nu(fg) = g(z)\nu(f) + f(z)\nu(g)$.
- By definition, $\dfrac{\partial f}{\partial x_i^\alpha}(z) = \dfrac{\partial(f \circ \varphi_\alpha)}{\partial x_i^\alpha}(\varphi_\alpha(z))$ and similarly for the y_i^αs.
- Hence, $\dfrac{\partial}{\partial x_i^\alpha}(z)$ is a tangent vector at z and

$$\left\{ \frac{\partial}{\partial x_1^\alpha}(z), \frac{\partial}{\partial y_1^\alpha}(z), \ldots, \frac{\partial}{\partial x_n^\alpha}(z), \frac{\partial}{\partial y_n^\alpha}(z) \right\}$$

is a real basis of T_zM (exercise).

MANIFOLDS

- Complexify T_zM, that is, $T_zM^{\mathbb{C}} = T_zM \otimes \mathbb{C}$ (= simply allow multiplication by complex numbers). This is a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} T_zM^{\mathbb{C}} = 2n$. For $z \in U_\alpha$, choose for $T_zM^{\mathbb{C}}$ the basis

$$\left\{ \frac{\partial}{\partial z_1^\alpha}(z), \frac{\partial}{\partial \bar{z}_1^\alpha}(z), \dots, \frac{\partial}{\partial z_n^\alpha}(z), \frac{\partial}{\partial \bar{z}_n^\alpha}(z) \right\}$$

where $\dfrac{\partial}{\partial z_k^\alpha}(z) = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_k^\alpha}(z) - \mathrm{i}\dfrac{\partial}{\partial y_k^\alpha}(z) \right)$ and
$\dfrac{\partial}{\partial \bar{z}_k^\alpha}(z) = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_k^\alpha}(z) + \mathrm{i}\dfrac{\partial}{\partial y_k^\alpha}(z) \right)$

MANIFOLDS

- Let's examine changes of coordinates. Set $\widetilde{\Theta}_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. Write $\widetilde{\Theta}_{\alpha\beta}(x_1, y_1, \ldots, x_n, y_n) = (u_1, v_1, \ldots, u_n, v_n)$ (real coordinates). The derivative is given by the matrix

MANIFOLDS

- Let's examine changes of coordinates. Set $\widetilde{\Theta}_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. Write $\widetilde{\Theta}_{\alpha\beta}(x_1, y_1, \ldots, x_n, y_n) = (u_1, v_1, \ldots, u_n, v_n)$ (real coordinates). The derivative is given by the matrix
- $$D\widetilde{\Theta}_{\alpha\beta} = \begin{pmatrix} \dfrac{\partial(u_1, v_1)}{\partial(x_1, y_1)} & \cdots & \dfrac{\partial(u_n, v_n)}{\partial(x_n, y_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial(u_n, v_n)}{\partial(x_1, y_1)} & \cdots & \dfrac{\partial(u_n, v_n)}{\partial(x_n, y_n)} \end{pmatrix}$$

MANIFOLDS

- Now write $\widetilde{\Theta}_{\alpha\beta} = (\widetilde{\Theta}_1, \ldots, \widetilde{\Theta}_n)$ where $\widetilde{\Theta}_j = u_j + \mathrm{i} v_j$.

MANIFOLDS

- Now write $\widetilde{\Theta}_{\alpha\beta} = (\widetilde{\Theta}_1, \ldots, \widetilde{\Theta}_n)$ where $\widetilde{\Theta}_j = u_j + \mathrm{i} v_j$.
- Changing from the basis

$$\left\{ \frac{\partial}{\partial x_1}(z), \frac{\partial}{\partial y_1}(z), \ldots, \frac{\partial}{\partial x_n}(z), \frac{\partial}{\partial y_n}(z) \right\}$$

to the basis

$$\left\{ \frac{\partial}{\partial z_1}(z), \frac{\partial}{\partial \bar{z}_1}(z), \ldots, \frac{\partial}{\partial z_n}(z), \frac{\partial}{\partial \bar{z}_n}(z) \right\}$$

MANIFOLDS

- and finally changing from the basis

$$\left\{ \frac{\partial}{\partial z_1}(z), \frac{\partial}{\partial \bar{z}_1}(z), \ldots, \frac{\partial}{\partial z_n}(z), \frac{\partial}{\partial \bar{z}_n}(z) \right\}$$

to the basis

$$\left\{ \frac{\partial}{\partial z_1}(z), \ldots, \frac{\partial}{\partial z_n}(z), \frac{\partial}{\partial \bar{z}_1}(z), \ldots, \frac{\partial}{\partial \bar{z}_n}(z) \right\}$$

MANIFOLDS

- the derivative $D\widetilde{\Theta}_{\alpha\beta}$ has the matrix

$$D\widetilde{\Theta}_{\alpha\beta} = \begin{pmatrix} \Theta_{\alpha\beta} & 0 \\ 0 & \bar{\Theta}_{\alpha\beta} \end{pmatrix}$$

where

$$\Theta_{\alpha\beta} = \left(\frac{\partial \widetilde{\Theta}_i}{\partial z_j} \right)_{1 \leq i,j \leq n}$$

MANIFOLDS

- the derivative $D\widetilde{\Theta}_{\alpha\beta}$ has the matrix

$$D\widetilde{\Theta}_{\alpha\beta} = \begin{pmatrix} \Theta_{\alpha\beta} & 0 \\ 0 & \bar{\Theta}_{\alpha\beta} \end{pmatrix}$$

where

$$\Theta_{\alpha\beta} = \left(\frac{\partial \widetilde{\Theta}_i}{\partial z_j} \right)_{1 \leq i,j \leq n}$$

- Hence, $\det D\widetilde{\Theta}_{\alpha\beta} = \det \Theta_{\alpha\beta} \det \bar{\Theta}_{\alpha\beta} = |\det \Theta_{\alpha\beta}|^2 > 0$ and complex manifolds are born orientable.

MANIFOLDS

- We use this last basis to decompose $T_z M^{\mathbb{C}}$ into 2 subspaces.

MANIFOLDS

- We use this last basis to decompose $T_z M^{\mathbb{C}}$ into 2 subspaces.
- $$T'_z M = \left\langle \frac{\partial}{\partial z_1}(z), \dots \frac{\partial}{\partial z_n}(z) \right\rangle_{\mathbb{C}}$$

  the holomorphic tangent space and

MANIFOLDS

- We use this last basis to decompose $T_z M^{\mathbb{C}}$ into 2 subspaces.
- $$T_z' M = \left\langle \frac{\partial}{\partial z_1}(z), \ldots \frac{\partial}{\partial z_n}(z) \right\rangle_{\mathbb{C}}$$

  the holomorphic tangent space and
- $$T_z'' M = \left\langle \frac{\partial}{\partial \bar{z}_1}(z), \ldots, \frac{\partial}{\partial \bar{z}_n}(z) \right\rangle_{\mathbb{C}}$$

  the anti-holomorphic tangent space.

MANIFOLDS

- We use this last basis to decompose $T_zM^{\mathbb{C}}$ into 2 subspaces.
- $$T_z'M = \left\langle \frac{\partial}{\partial z_1}(z), \ldots \frac{\partial}{\partial z_n}(z) \right\rangle_{\mathbb{C}}$$
  the holomorphic tangent space and
- $$T_z''M = \left\langle \frac{\partial}{\partial \bar{z}_1}(z), \ldots, \frac{\partial}{\partial \bar{z}_n}(z) \right\rangle_{\mathbb{C}}$$
  the anti-holomorphic tangent space.
- So $T_zM^{\mathbb{C}} = T_z'M \oplus T_z''M$.

MANIFOLDS

- Let's restrict now to real manifolds (just to avoid heavy notation). So suppose M is a real manifold of dimension m and class C^k.

MANIFOLDS

- Let's restrict now to real manifolds (just to avoid heavy notation). So suppose M is a real manifold of dimension m and class C^k.
- A tangent vector ν at $a \in M$ acts on functions and $df_a.\nu = \nu(f) = \sum_1^m \nu_j \partial f/\partial x_j(a)$.

MANIFOLDS

- Let's restrict now to real manifolds (just to avoid heavy notation). So suppose M is a real manifold of dimension m and class C^k.
- A tangent vector ν at $a \in M$ acts on functions and $df_a.\nu = \nu(f) = \sum_1^m \nu_j \, \partial f/\partial x_j(a)$.
- Since $dx_j.\nu = \nu_j$ we have $df = \sum_1^m (\partial f/\partial x_j) dx_j$. This means that the dual basis of $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_m\}$ is $\{dx_1, \ldots, dx_m\}$. The dual space $T_x^* M$ of $T_x M$ is called the cotangent space.

MANIFOLDS

- Let's restrict now to real manifolds (just to avoid heavy notation). So suppose M is a real manifold of dimension m and class C^k.
- A tangent vector ν at $a \in M$ acts on functions and $df_a.\nu = \nu(f) = \sum_1^m \nu_j \partial f/\partial x_j(a)$.
- Since $dx_j.\nu = \nu_j$ we have $df = \sum_1^m (\partial f/\partial x_j) dx_j$. This means that the dual basis of $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_m\}$ is $\{dx_1, \ldots, dx_m\}$. The dual space $T_x^* M$ of $T_x M$ is called the cotangent space.
- The disjoint unions $TM = \cup_{x\in M} T_x M$ and $T^*M = \cup_{x\in M} T_x^* M$ are the tangent and cotangent bundles of M.

MANIFOLDS

- Consider the real algebra Λ^* generated by $dx_1, \ldots, dx_n$ with the relations $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for $i \neq j$. As a vector space this algebra has basis:

$$1, dx_i, dx_i \wedge dx_j (i < j), dx_i \wedge dx_j \wedge dx_k (i < j < k), ...$$

MANIFOLDS

- Consider the real algebra Λ^* generated by $dx_1, \ldots, dx_n$ with the relations $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for $i \neq j$. As a vector space this algebra has basis:

$$1, dx_i, dx_i \wedge dx_j (i < j), dx_i \wedge dx_j \wedge dx_k (i < j < k), \ldots$$

- Differential forms of class C^k on $\mathbb{R}^n$ are elements of

$$C^k(\mathbb{R}^n, \mathbb{R}) \otimes_{\mathbb{R}} \Lambda^*.$$

MANIFOLDS

- Consider the real algebra Λ^* generated by $dx_1, \ldots, dx_n$ with the relations $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for $i \neq j$. As a vector space this algebra has basis:

$$1, dx_i, dx_i \wedge dx_j (i < j), dx_i \wedge dx_j \wedge dx_k (i < j < k), ...$$

- Differential forms of class C^k on $\mathbb{R}^n$ are elements of

$$C^k(\mathbb{R}^n, \mathbb{R}) \otimes_{\mathbb{R}} \Lambda^*.$$

- The same applies locally to manifolds.

MANIFOLDS

- Hence, a differential form of degree q, or a q-form on M, is a map u on M with values $u(x) \in \Lambda^q T_x^* M$. In an open coordinate patch $U \subset M$, $u(x)$ can be written

$$u(x) = \sum_{|I|=q} u_{|I|}(x) dx_{|I|}$$

where $I = (i_1, \ldots, i_q)$ is a multi-index, $i_1 < \cdots < i_q$ and $dx_{|I|} = dx_{i_1} \wedge \cdots \wedge dx_{i_q}$.

MANIFOLDS

- Hence, a differential form of degree q, or a q-form on M, is a map u on M with values $u(x) \in \Lambda^q T_x^* M$. In an open coordinate patch $U \subset M$, $u(x)$ can be written

$$u(x) = \sum_{|I|=q} u_{|I|}(x) dx_{|I|}$$

where $I = (i_1, \ldots, i_q)$ is a multi-index, $i_1 < \cdots < i_q$ and $dx_{|I|} = dx_{i_1} \wedge \cdots \wedge dx_{i_q}$.
- For all $0 \leq q \leq m$, $0 \leq k \leq \infty$, $A_k^q(M)$ denotes the space of C^k q-forms on M, i.e., forms with $u_{|I|}$ functions of class C^k.

MANIFOLDS

- The exterior derivative is the operator

$$d : A_k^q(M) \longrightarrow A_{k-1}^{q+1}(M)$$

defined locally by

$$du = \sum du_{|I|} \wedge dx_{|I|}$$

MANIFOLDS

- The exterior derivative is the operator

$$d : A_k^q(M) \longrightarrow A_{k-1}^{q+1}(M)$$

defined locally by

$$du = \sum du_{|I|} \wedge dx_{|I|}$$

- It satisfies $d(u \wedge v) = du \wedge v + (-1)^{\deg u} u \wedge dv$ and $d^2 = 0$.

MANIFOLDS

- The exterior derivative is the operator

$$d : A_k^q(M) \longrightarrow A_{k-1}^{q+1}(M)$$

defined locally by

$$du = \sum du_{|I|} \wedge dx_{|I|}$$

- It satisfies $d(u \wedge v) = du \wedge v + (-1)^{\deg u} u \wedge dv$ and $d^2 = 0$.
- A form u is closed if $du = 0$ and exact if $u = dv$ for some form v.

MANIFOLDS

- Finally, a cohomological complex $K^\bullet = \bigoplus_{q\in\mathbb{Z}} K^q$ is a collection of modules over a ring, endowed with differentials, that is, linear maps $d^q : K^q \longrightarrow K^{q+1}$ satisfying $d^{q+1} \circ d^q = 0$.

MANIFOLDS

- Finally, a cohomological complex $K^\bullet = \bigoplus_{q\in\mathbb{Z}} K^q$ is a collection of modules over a ring, endowed with differentials, that is, linear maps $d^q : K^q \longrightarrow K^{q+1}$ satisfying $d^{q+1} \circ d^q = 0$.
- The associated cocycle, coboundary and cohomology modules are defined respectively by

$$
\begin{aligned}
&Z^q(K^\bullet) = \ker d^q\,, \qquad Z^q(K^\bullet) \subset K^q\\
&B^q(K^\bullet) = \mathrm{Im}\, d^{q-1}\,, \qquad B^q(K^\bullet) \subset Z^q(K^\bullet) \subset K^q\\
&H^q(K^\bullet) = Z^q(K^\bullet)/B^q(K^\bullet)
\end{aligned}
$$

MANIFOLDS

- If M is a real C^∞ manifold, the De Rham complex of M is the cohomological complex

$$A^\bullet_\infty(M) = \bigoplus_{q \geq 0} A^q_\infty(M)$$

with differential d, the exterior derivative.

MANIFOLDS

- If M is a real C^∞ manifold, the De Rham complex of M is the cohomological complex

$$A^\bullet_\infty(M) = \bigoplus_{q\geq 0} A^q_\infty(M)$$

with differential d, the exterior derivative.

- We denote its cohomology groups by $H^q_{DR}(M,\mathbb{R}) = Z^q(M,\mathbb{R})/B^q(M,\mathbb{R})$.

MANIFOLDS

- A real manifold M is orientable in case it admits an atlas with all transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ with positive jacobian determinant. Suppose M is oriented by such an atlas. If $u(x) = g(x_1, \ldots, x_n)\, dx_1 \wedge \cdots \wedge dx_m$ is a continuous m-form on M, with $m = \dim_{\mathbb{R}} M$, with compact support in a coordinate system, define $\int_M u = \int_{\mathbb{R}^m} f\, dx_1 \ldots dx_m$. This is independent of the coordinate system (orientability). If u has compact support, we extend this definition of $\int_M u$ by means of a partition of unity.

MANIFOLDS

- Now, if $K \subset M$ is a compact set with piecewise C^1 boundary ∂K, it's possible to give an orientation to ∂K in such a way that for any differential form of class C^1 and of degree $m-1$ we have

$$\int_{\partial K} u = \int_K du.$$

This is Stokes formula.

CURRENTS

- A C^∞ p-form ω on $U \subset \mathbb{C}^n$ is given by a sum of terms of the types $f_I dx_I$, $g_J dy_J$ and $h_K d(x,y)_K$, where $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, $dy_J = dy_{j_1} \wedge dy_{j_2} \wedge \cdots \wedge dy_{j_p}$, $d(x,y)_K$ is a product of p-forms of types dx_i and dy_j, and f_I, g_J, h_K are smooth complex valued functions. Now, $dx_i = (1/2)(dz_i + d\bar{z}_i)$ and $dy_i = (1/2\mathrm{i})(dz_i - d\bar{z}_i)$. Expressing the terms in ω by using dz_i and $d\bar{z}_i$ we arrive at

$$\omega = \sum k_{i_1,\ldots,i_r,j_1,\ldots,j_s} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s},$$

which we abbreviate as $\omega = \sum k_{I,J}\, dz_I \wedge d\bar{z}_J$. We say that each term of this sum is a p-form of type (r,s), $r+s=p$.

CURRENTS

- It follows that a p-form ω has a unique expression as a sum
$$\omega = \omega^{(p,0)} + \omega^{(p-1,1)} + \cdots + \omega^{(0,p)},$$
where $\omega^{(r,s)}$ is of type (r,s).

CURRENTS

- It follows that a p-form ω has a unique expression as a sum
$$\omega = \omega^{(p,0)} + \omega^{(p-1,1)} + \cdots + \omega^{(0,p)},$$
where $\omega^{(r,s)}$ is of type (r,s).
- Let $A^0(U)$ be the $\mathbb{C}$-algebra $C^\infty(U,\mathbb{C})$ and $A^p(U)$ the $A^0(U)$-module of C^∞ complex p-forms on U. The decomposition above induces a decomposition
$$A^p(U) = A^{(p,0)}(U) \oplus A^{(p-1,1)}(U) \oplus \cdots \oplus A^{(0,p)}(U).$$
We have the exterior differential $d : A^p(U) \to A^{p+1}(U)$.

CURRENTS

- For $f \in A^0(U)$

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

Define, on the level of functions,

$$\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i \quad \text{and} \quad \bar{\partial} f = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

CURRENTS

- On the level of forms, if

$$\omega^{(r,s)} = \sum k_{i_1,\ldots,i_r,j_1,\ldots,j_s} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s},$$

we let

$$\partial\omega^{(r,s)} = \sum \partial k_{i_1,\ldots,i_r,j_1,\ldots,j_s} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s}$$

a form of type $(r+1, s)$ and

$$\bar{\partial}\omega^{(r,s)} = \sum \bar{\partial} k_{i_1,\ldots,i_r,j_1,\ldots,j_s} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s}$$

of type $(r, s+1)$.

CURRENTS

- We are left with

$$d\omega^{(r,s)} \;=\; \partial\omega^{(r,s)} + \overline{\partial}\omega^{(r,s)}.$$

For an arbitrary p-form $\omega = \sum_{r+s=p} \omega^{(r,s)}$, we put

$$\partial\omega \;=\; \sum_{r+s=p} \partial\omega^{(r,s)} \;\text{ and }\; \overline{\partial}\omega \;=\; \sum_{r+s=p} \overline{\partial}\omega^{(r,s)}.$$

CURRENTS

- We are left with

$$d\omega^{(r,s)} = \partial\omega^{(r,s)} + \overline{\partial}\omega^{(r,s)}.$$

For an arbitrary p-form $\omega = \sum_{r+s=p} \omega^{(r,s)}$, we put

$$\partial\omega = \sum_{r+s=p} \partial\omega^{(r,s)} \text{ and } \overline{\partial}\omega = \sum_{r+s=p} \overline{\partial}\omega^{(r,s)}.$$

- It follows that $d = \partial + \overline{\partial}$ and the following properties hold (exercise):

$$\partial(\omega^p \wedge \eta) = \partial\omega^p \wedge \eta + (-1)^p \omega^p \wedge \partial\eta,$$

$$\overline{\partial}(\omega^p \wedge \eta) = \overline{\partial}\omega^p \wedge \eta + (-1)^p \omega^p \wedge \overline{\partial}\eta.$$

CURRENTS

- Moreover, (exercise)

$$\partial\partial\omega^{(r,s)} + \overline{\partial}\partial\omega^{(r,s)} + \partial\overline{\partial}\omega^{(r,s)} + \overline{\partial}\,\overline{\partial}\omega^{(r,s)} \;=\; dd\omega^{(r,s)} \;=\; 0.$$

By comparing the form types in the above summation we conclude that

$$\partial^2 = \partial\,\partial \;=\; 0 \;,\;\; \partial\,\overline{\partial} + \overline{\partial}\,\partial \;=\; 0 \;,\;\; \overline{\partial}^2 = \overline{\partial}\,\overline{\partial} \;=\; 0.$$

CURRENTS

- A $(p,0)$-form $\omega^{(p,0)} = \sum f_{i_1,\ldots,i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ is *holomorphic* if the coefficients $f_{i_1,\ldots,i_p}$ are holomorphic functions. In this case,

$$\overline{\partial}\omega \;=\; \sum \overline{\partial} f_{i_1,\ldots,i_p} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \;=\; 0.$$

Conversely, if $\overline{\partial}\omega^{(p,0)} = 0$, then ω has holomorphic coefficients. For holomorphic forms we have $\partial\omega = d\omega$.

CURRENTS

- Let $A^q_{\infty,c}(\mathbb{R}^n) = A^q_c(\mathbb{R}^n)$ be the space of C^∞ q-forms on $\mathbb{R}^n$ with compact support.

CURRENTS

- Let $A^q_{\infty,c}(\mathbb{R}^n) = A^q_c(\mathbb{R}^n)$ be the space of C^∞ q-forms on $\mathbb{R}^n$ with compact support.

- Definition

  The topological dual of $A^{n-q}_c(\mathbb{R}^n)$ is the space of currents of degree q, denoted $\mathcal{D}^q(\mathbb{R}^n)$. This means that $\mathcal{D}^q(\mathbb{R}^n)$ is the space of continuous linear forms T on $A^{n-q}_c(\mathbb{R}^n)$.

CURRENTS

- Let $A^q_{\infty,c}(\mathbb{R}^n) = A^q_c(\mathbb{R}^n)$ be the space of C^∞ q-forms on $\mathbb{R}^n$ with compact support.

- **Definition**
  The topological dual of $A^{n-q}_c(\mathbb{R}^n)$ is the space of currents of degree q, denoted $\mathcal{D}^q(\mathbb{R}^n)$. This means that $\mathcal{D}^q(\mathbb{R}^n)$ is the space of continuous linear forms T on $A^{n-q}_c(\mathbb{R}^n)$.

  - Example 1. Let $L^q_{loc}(\mathbb{R}^n)$ be the space of q-forms $u(x) = \sum_{|I|=q} u_{|I|}(x)dx_{|I|}$ whose coefficients $u_{|I|}(x)$ are locally integrable.

$$T_u(\phi) = \int_{\mathbb{R}^n} u \wedge \phi\,, \qquad \phi \in A^{n-q}_c(\mathbb{R}^n)$$

is the degree q current associated to u.

CURRENTS

- Example 2. Let Γ be a piecewise smooth oriented n-q chain in $\mathbb{R}^n$. Then

$$T_\Gamma(\phi) = \int_\Gamma \phi\,, \qquad \phi \in A_c^{n-q}(\mathbb{R}^n)$$

is the current in $\mathcal{D}^q(\mathbb{R}^n)$ defined by Γ.
This illustrates the concept of support: the $supp(T)$ of the current T is the smallest closed set S such that $T(\phi) = 0$ for all $\phi \in A_c^{n-q}(\mathbb{R}^n \setminus S)$. In the above case $supp(T_\Gamma) = \Gamma$.

CURRENTS

- The exterior derivative induces an operator

$$d : \mathcal{D}^q(\mathbb{R}^n) \longrightarrow \mathcal{D}^{q+1}(\mathbb{R}^n)$$

which, by definition, is:

$$(dT)(\phi) = (-1)^{q+1}T(d\phi)\,, \qquad \phi \in A_c^{n-q-1}(\mathbb{R}^n).$$

and it satisfies $d^2 = 0$. This is the beginning of residue theory.

CURRENTS

- The exterior derivative induces an operator

$$d : \mathcal{D}^q(\mathbb{R}^n) \longrightarrow \mathcal{D}^{q+1}(\mathbb{R}^n)$$

which, by definition, is:

$$(dT)(\phi) = (-1)^{q+1} T(d\phi)\,, \qquad \phi \in A_c^{n-q-1}(\mathbb{R}^n).$$

and it satisfies $d^2 = 0$. This is the beginning of residue theory.

- In example 1, by Stokes,

$$(dT_u)(\phi) = (-1)^{q+1} \int_{\mathbb{R}^n} u \wedge d\phi =$$
$$= -\int_{\mathbb{R}^n} d(u \wedge \phi) + \int_{\mathbb{R}^n} du \wedge \phi =$$
$$= T_{du}(\phi).$$

CURRENTS

- In example 2, by Stokes again,

$$
\begin{aligned}
(dT_\Gamma)(\phi) = (-1)^{q+1} \int_\Gamma d\phi = \\
= (-1)^{q+1} \int_{\partial\Gamma} \phi = \\
= (-1)^{q+1} T_{\partial\Gamma}(\phi).
\end{aligned}
$$

CURRENTS

- Let $\omega \in L^q_{loc}(\mathbb{R}^n)$ be C^∞ outside a closed set S. Suppose that $d\omega$ on $\mathbb{R}^n \setminus S$ extends to a locally integrable form on $\mathbb{R}^n$. The RESIDUE is the current defined by

$$dT_\omega - T_{d\omega} = Res(\omega).$$

We have $supp\, Res(\omega) \subset S$.

CURRENTS

- In $\mathbb{C}$, consider the Cauchy kernel

$$\kappa = \frac{1}{2\pi \mathrm{i}} \frac{dz}{z}.$$

Then, $\kappa \in L^{(1,0)}_{loc}(\mathbb{C})$ and is C^∞ on $\mathbb{C} \setminus \{0\}$, $d\kappa = \overline{\partial}\kappa = 0$ on $\mathbb{C} \setminus \{0\}$ and by the smooth version of Cauchy's formula, for $\phi \in C^\infty_c(\mathbb{C})$

$$\phi(0) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}^2} \frac{\partial \phi(z)}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z}.$$

CURRENTS

- In $\mathbb{C}$, consider the Cauchy kernel

$$\kappa = \frac{1}{2\pi\mathrm{i}}\frac{dz}{z}.$$

Then, $\kappa \in L^{(1,0)}_{loc}(\mathbb{C})$ and is C^∞ on $\mathbb{C}\setminus\{0\}$, $d\kappa = \overline{\partial}\kappa = 0$ on $\mathbb{C}\setminus\{0\}$ and by the smooth version of Cauchy's formula, for $\phi \in C^\infty_c(\mathbb{C})$

$$\phi(0) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}^2}\frac{\partial\phi(z)}{\partial\bar{z}}\frac{dz\wedge d\bar{z}}{z}.$$

- Hence $T_{d\kappa} = 0$ and $dT_\kappa = \overline{\partial}T_\kappa$. But this reads $(\overline{\partial}T_\kappa)(\phi) = \phi(0) = \delta_0(\phi)$ and $Res(\kappa) = \delta_0$, the Dirac function.

CURRENTS

- This can be generalized to $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by means of the Bochner-Martinelli kernel.

CURRENTS

- This can be generalized to $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by means of the Bochner-Martinelli kernel.
- We start with a kernel in $\mathbb{C}^n \times \mathbb{C}^n$, which is the complex analogue of the *Newtonian potential* in $\mathbb{R}^n \times \mathbb{R}^n$:

$$G(w,z) = \begin{cases} -\dfrac{1}{2\pi} \log |w-z|^2 & \text{for } n = 1 \\[2ex] \dfrac{(n-2)!}{2\pi^n} |w-z|^{2-2n} & \text{for } n \geq 2 \end{cases}$$

CURRENTS

- In what follows, w will denote the variable of integration and z will be a parameter and we let

$$\alpha_{2n-1} = \frac{2\pi^n}{(n-1)!} \quad \text{and} \quad \Lambda = |w - z|^2.$$

Notice that, since the area of the sphere $S_R^{2n-1} \subset \mathbb{C}^n$ of radius R is $\alpha_{2n-1} R^{2n-1}$, α_{2n-1} is just the area of the unit sphere S_1^{2n-1}.

CURRENTS

- The Bochner-Martinelli kernel (for functions) is the double form

$$K(w,z) = -*\partial_w G(w,z)$$

of type $(n, n-1)$ in w and type $(0,0)$ in z.
$K(w,z)$ is represented by the form

$$K = \frac{(n-1)!}{(2\pi \mathrm{i})^n |w-z|^{2n}} \sum_{i=1}^{n} (\bar{w}_i - \bar{z}_i)\, dw_i \wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right).$$

CURRENTS

- Set $n = 1$ to get the Cauchy kernel

$$\kappa = \frac{1}{2\pi \mathrm{i}} \frac{dw}{w - z}.$$

CURRENTS

- Set $n = 1$ to get the Cauchy kernel

$$\kappa = \frac{1}{2\pi \mathrm{i}} \frac{dw}{w - z}.$$

- $\overline{\partial}_w K(w, z) = 0$ on $\mathbb{C}^n \times \mathbb{C}^n \setminus \{w = z\}$.

CURRENTS

- Set $n = 1$ to get the Cauchy kernel

$$\kappa = \frac{1}{2\pi \mathrm{i}} \frac{dw}{w - z}.$$

- $\overline{\partial}_w K(w, z) = 0$ on $\mathbb{C}^n \times \mathbb{C}^n \setminus \{w = z\}$.
- K normalizes the area of spheres, more precisely: let $B_\epsilon(z)$ denote the euclidean ball centered at z and with radius ϵ. Then,

$$\int_{\partial B_\epsilon(z)} K(w, z) = 1$$

for all $z \in \mathbb{C}^n$ and for all $\epsilon > 0$.

CURRENTS

- Finally we have the Bochner-Martinelli integral formula

Theorem

Let $U \subset \mathbb{C}^n$ be a limited domain whose boundary ∂U is a smooth manifold. Suppose $f : \bar{U} \to \mathbb{C}$ is continuous and f is holomorphic in U. Then,

$$\int_{\partial U} f(w)\, K(w,z) = \begin{cases} f(z) & \text{for } z \in U \\ 0 & \text{for } z \notin U. \end{cases}$$

CURRENTS

- Proceeding verbatim as we did in the case of the Cauchy kernel in $\mathbb{C}$, we have that

$$\overline{\partial}_w T_K = \delta_z$$

and

$$Res(K) = \delta_z.$$

CURRENTS

- A current $T \in \mathcal{D}^q(\mathbb{R}^n)$ may be considered as a differential form whose coefficients T_I are distributions:

$$T = \sum_{|I|=q} T_I \, dx_I$$

These distributions are defined by $T_I(\phi) = \pm T(\phi dx_{I_0})$ where $*dx_I = \pm dx_{I_0}$. The smoothing

$$T_\epsilon = \sum_{|I|=q} (T_I)_\epsilon \, dx_I$$

satisfies

$$dT_\epsilon = d(T_\epsilon).$$

CURRENTS

- If M is a complex manifold, the currents $\mathcal{D}^{(p,p)}(M)$ of type (p,p) are the continuous linear forms on $A_c^{n-p,n-p}(M)$. A (p,p)-current is real if $T = \overline{T}$, that is, $\overline{T(\phi)} = T(\overline{\phi})$ for all $\phi \in A_c^{n-p,n-p}(M)$.

CURRENTS

- If M is a complex manifold, the currents $\mathcal{D}^{(p,p)}(M)$ of type (p,p) are the continuous linear forms on $A_c^{n-p,n-p}(M)$. A (p,p)-current is real if $T = \overline{T}$, that is, $\overline{T(\phi)} = T(\overline{\phi})$ for all $\phi \in A_c^{n-p,n-p}(M)$.
- A real current is positive if

$$\mathrm{i}^{p(p-1)/2} T(\eta \wedge \overline{\eta}) \geq 0, \qquad \eta \in A_c^{n-p,0}(M).$$

The positivity of T implies that it has order 0 in the sense of distributions and hence defines a measure (positive).

CURRENTS

- An important example is: if $Z \subset M$ is a codimension p analytic subvariety and Z_{reg} is the set of smooth points of Z, then the map

$$T_Z(\phi) = \int_{Z_{reg}} \phi, \qquad \phi \in A_c^{n-p,n-p}(M)$$

defines a closed positive current, which is the fundamental class of Z via the isomorphism

$$H^\bullet_{DR}(M) \approx H^\bullet(\mathcal{D}^\bullet(M), d).$$

CURRENTS

- A C^{∞} $(1,1)$-form

$$\omega = \frac{\mathrm{i}}{2} \sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j$$

is real if $\overline{h_{ij}} = h_{ji}$, positive if the matrix h_{ij} is positive definite and closed when the associated hermitian metric $ds^2 = \sum_{i,j} h_{ij} dz_i d\bar{z}_j$ is Kähler.

CURRENTS

- A real function $\phi \in L^1_{loc}(M)$ is plurisubharmonic in case $\mathrm{i}\partial\overline{\partial}\phi$ is a positive $(1,1)$-current (derivatives are in the sense of distributional derivatives). There is the $\partial\overline{\partial}$-Poincaré lemma:

CURRENTS

- A real function $\phi \in L^1_{loc}(M)$ is plurisubharmonic in case $\mathrm{i}\partial\overline{\partial}\phi$ is a positive $(1,1)$-current (derivatives are in the sense of distributional derivatives). There is the $\partial\overline{\partial}$-Poincaré lemma:
- Let T be a closed, positive $(1,1)$-current. Then, locally,

$$T = \mathrm{i}\partial\overline{\partial}\phi$$

for a real plurisubharmonic function ϕ, uniquely determined up to addition of the real part of a holomorphic function.

CURRENTS

- Now we specialize to $M = \mathbb{P}^n_{\mathbb{C}}$, the complex projective space of dimension n.

CURRENTS

- Now we specialize to $M = \mathbb{P}^n_{\mathbb{C}}$, the complex projective space of dimension n.
- A foliation of dimension 1, $\mathcal{F}$, on $\mathbb{P}^n_{\mathbb{C}}$ is, vaguely saying, the set of orbits of a rational vector field. In a precise way, it is generated by a nontrivial holomorphic section

$$s \in H^0(\mathbb{P}^n_{\mathbb{C}}, \Theta_{\mathbb{P}^n_{\mathbb{C}}} \otimes \mathcal{O}(d-1))$$

where d is an integer, the degree of $\mathcal{F}$. We suppose that the singular set of $\mathcal{F}$ (the zeros of s) has codimension at least two, which means that s is uniquely determined up to a multiplicative constant. This tells us that the space $Fol(d,n)$ is Zariski open in $\mathbb{P}(H^0(\mathbb{P}^n_{\mathbb{C}}, \Theta_{\mathbb{P}^n_{\mathbb{C}}} \otimes \mathcal{O}(d-1)))$.

CURRENTS

- Write $S(\mathcal{F})$ for the singular set of $\mathcal{F}$. Outside $S(\mathcal{F})$ we have a nonsingular foliation $\mathcal{F}_{reg} = \mathcal{F}_{|\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})}$. For nonsingular foliations there is a notion of invariant measures which we describe briefly:

CURRENTS

- Write $S(\mathcal{F})$ for the singular set of $\mathcal{F}$. Outside $S(\mathcal{F})$ we have a nonsingular foliation $\mathcal{F}_{reg} = \mathcal{F}_{|\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})}$. For nonsingular foliations there is a notion of invariant measures which we describe briefly:
- Let $\mathfrak{D}$ be a finite union of closed discs transverse to the foliation, whose interiors meet every leaf. If a path on one leaf connects 2 points x and y in $\mathfrak{D}$, with y in the interior, the foliation determines a germ of homeomorphism from a neighborhood of x in $\mathfrak{D}$ into one of y in $\mathfrak{D}$.

CURRENTS

- A transversal invariant measure for the foliation is a non-negative measure of finite mass on $\mathfrak{D}$ which is compatible with all the germs of homeomorphisms determined by the foliation.

CURRENTS

- A transversal invariant measure for the foliation is a non-negative measure of finite mass on $\mathfrak{D}$ which is compatible with all the germs of homeomorphisms determined by the foliation.
- D. Sullivan showed that: There exists a natural bijective correspondence between invariant measures for $\mathcal{F}_{reg}$ and *invariant* closed positive $(1,1)$-currents (recall that $\mathcal{F}$ has complex dimension 1, hence real dimension 2).

CURRENTS

- A transversal invariant measure for the foliation is a non-negative measure of finite mass on $\mathfrak{D}$ which is compatible with all the germs of homeomorphisms determined by the foliation.
- D. Sullivan showed that: There exists a natural bijective correspondence between invariant measures for $\mathcal{F}_{reg}$ and *invariant* closed positive $(1,1)$-currents (recall that $\mathcal{F}$ has complex dimension 1, hence real dimension 2).
- A closed positive current T on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$ is *invariant* by $\mathcal{F}$ if $T(\omega) = 0$ for every 2-form ω which vanishes on the leaves of $\mathcal{F}$.

CURRENTS

- Let us exemplify this:

CURRENTS

- Let us exemplify this:
- Locally, take coordinates $(z_1, \ldots, z_n)$ on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$, such that $\mathcal{F}$ is generated by $\partial/\partial z_1$. Let

$$\omega_i = dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n\,,\ 1 \leq n.$$

CURRENTS

- Let us exemplify this:
- Locally, take coordinates $(z_1, \ldots, z_n)$ on $\mathbb{P}^n_{\mathbb{C}} \setminus S(\mathcal{F})$, such that $\mathcal{F}$ is generated by $\partial/\partial z_1$. Let

$$\omega_i = dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n \,, \; 1 \leq n.$$

- The positive current T can be locally written in the form

$$T = \sum_{i,j=1}^{n} f_{ij} \, \mathrm{i} \, \omega_i \wedge \overline{\omega}_j$$

where f_{ij} are complex valued measures.

CURRENTS

- To say that T is $\mathcal{F}$-invariant means $T \wedge dz_i = 0$ and $T \wedge d\bar{z}_i = 0$ for all $i \neq 1$. This gives $f_{ij} = 0$ for $(i, j) \neq (1, 1)$ and then

$$T = f_{11}\,\mathrm{i}\,\omega_1 \wedge \overline{\omega}_1.$$

CURRENTS

- To say that T is $\mathcal{F}$-invariant means $T \wedge dz_i = 0$ and $T \wedge d\bar{z}_i = 0$ for all $i \neq 1$. This gives $f_{ij} = 0$ for $(i, j) \neq (1, 1)$ and then

$$T = f_{11}\, \mathrm{i}\, \omega_1 \wedge \overline{\omega}_1.$$

- T closed means that its distributional derivatives along z_1 and $\bar{z}_1$ are 0. Hence, T does not depend on z_1 and projects to a positive measure on the local transversal $z_1 = 0$ (invariant by the holonomy).

CURRENTS

- To say that T is $\mathcal{F}$-invariant means $T \wedge dz_i = 0$ and $T \wedge d\bar{z}_i = 0$ for all $i \neq 1$. This gives $f_{ij} = 0$ for $(i,j) \neq (1,1)$ and then

$$T = f_{11}\,\mathrm{i}\,\omega_1 \wedge \overline{\omega}_1.$$

- T closed means that its distributional derivatives along z_1 and $\bar{z}_1$ are 0. Hence, T does not depend on z_1 and projects to a positive measure on the local transversal $z_1 = 0$ (invariant by the holonomy).
- Suppose now that the foliation has only isolated singularities, hence $S(\mathcal{F})$ is a finite number of points. This is the generic situation. In this case the closed positive current T can be extended to all of $\mathbb{P}^n_{\mathbb{C}}$ (Hartogs).

CURRENTS

- We want to comment on the following result of M. Brunella (2006):

CURRENTS

- We want to comment on the following result of M. Brunella (2006):

- **Theorem**

  Given $n \geq 2$ and $d \geq 2$, there exists an open and dense subset $\mathbf{U} \subset Fol(n, d)$ such that any $\mathcal{F} \in \mathbf{U}$ has no invariant measure.

CURRENTS

- We want to comment on the following result of M. Brunella (2006):

- Theorem
  Given $n \geq 2$ and $d \geq 2$, there exists an open and dense subset $\mathbf{U} \subset Fol(n, d)$ such that any $\mathcal{F} \in \mathbf{U}$ has no invariant measure.
  - Lins Neto and S. proved the following (1996):

CURRENTS

- We want to comment on the following result of M. Brunella (2006):

- **Theorem**
  Given $n \geq 2$ and $d \geq 2$, there exists an open and dense subset $\mathbf{U} \subset Fol(n, d)$ such that any $\mathcal{F} \in \mathbf{U}$ has no invariant measure.

- Lins Neto and S. proved the following (1996):

- **Theorem**
  Given $n \geq 2$ and $d \geq 2$, there exists an open and dense subset $\mathbf{U} \subset Fol(n, d)$ such that any $\mathcal{F} \in \mathbf{U}$ has no invariant algebraic curve.

CURRENTS

- We want to comment on the following result of M. Brunella (2006):

- Theorem
  Given $n \geq 2$ and $d \geq 2$, there exists an open and dense subset $\mathbf{U} \subset Fol(n, d)$ such that any $\mathcal{F} \in \mathbf{U}$ has no invariant measure.

  - Lins Neto and S. proved the following (1996):

- Theorem
  Given $n \geq 2$ and $d \geq 2$, there exists an open and dense subset $\mathbf{U} \subset Fol(n, d)$ such that any $\mathcal{F} \in \mathbf{U}$ has no invariant algebraic curve.

  - $\mathbf{U}$ is exactly the same set in both theorems.

CURRENTS

- **U** is the set of foliations with the following two properties: (i) all the singularities of $\mathcal{F}$ are hyperbolic and (ii) $\mathcal{F}$ has no invariant algebraic curve.

CURRENTS

- **U** is the set of foliations with the following two properties: (i) all the singularities of $\mathcal{F}$ are hyperbolic and (ii) $\mathcal{F}$ has no invariant algebraic curve.
- A singularity p of $\mathcal{F}$ is hyperbolic if around p the foliation is generated by a vector field whose linear part at has eigenvalues $\lambda_1, \ldots, \lambda_n$ such that

$$\frac{\lambda_i}{\lambda_j} \notin \mathbb{R}.$$

CURRENTS

- Brunella's proof runs as follows: he produces a residue theorem for currents which gives a relation between these residues and a global geometric object associated to the foliation:

CURRENTS

- Brunella's proof runs as follows: he produces a residue theorem for currents which gives a relation between these residues and a global geometric object associated to the foliation:
- $$c_1(\det N^*_{\mathcal{F}}).[T] = \sum_{p \in S(\mathcal{F}) \cap supp(T)} Res(\mathcal{F}, T, p)$$

  where $c_1(\det N^*_{\mathcal{F}}) \in H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{R})$ is the first Chern class of the conormal bundle of $\mathcal{F}$ and $[T] \in H_2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{R})$ id the homology class of T.

CURRENTS

- Then, assuming the foliation has no invariant algebraic curves and only hyperbolic singularities, it's shown that $Res(\mathcal{F}, T, p) = 0$. But this implies that $c_1(\det N^*_{\mathcal{F}}).[T] = 0$ which is absurd since $\det N^*_{\mathcal{F}} = \mathcal{O}(-n - d)$, a negative line bundle which has negative degree on any positive homology class, like $[T]$ for instance.